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A geometrical version of the Helmholtz conditions in time-dependent Lagrangian dynamics

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Abstract. Appropriate geometrical machinery for the study of time-dependent Lagrangian dynamics is developed. It is applied to the inverse problem of the calculus of variations, and a set of necessary and sufficient conditions for the existence of a Lagrangian are given, in terms of the existence of a 2-form with suitable properties, which are exactly equivalent to the Helmholtz conditions.

1. Introduction

The Helmholtz conditions are the conditions that must be satisfied by a non-singular multiplier matrix $(\alpha_{ab}(t, x, \dot{x}))$ in order that a given system of second-order ordinary differential equations

$$\ddot{x}^a = f^a(t, x, \dot{x}),$$

when written in the form

$$\alpha_{ab}\ddot{x}^b + \beta_a = 0 \qquad (\beta_a = -\alpha_{ab}f^b),$$

become the Euler–Lagrange equations for some Lagrangian function $L(t, x, \dot{x})$. We quote them in the form given by Sarlet (1982) and derived earlier by Douglas (1941). First, following Sarlet, we define functions A^a_b, B^a_b, Φ^a_b by

$$A^a_b = -\tfrac{1}{2}\partial f^a/\partial \dot{x}^b, \qquad B^a_b = -\partial f^a/\partial x^b, \qquad \Phi^a_b = B^a_b - A^c_b A^a_c - \Gamma(A^a_b),$$

where Γ stands for the differential operator (or vector field)

$$\partial/\partial t + \dot{x}^a\, \partial/\partial x^a + f^a\, \partial/\partial \dot{x}^a.$$

Then the necessary and sufficient conditions for the existence of a Lagrangian for the equations $\alpha_{ab}\ddot{x}^b + \beta_a = 0$ are that the functions α_{ab} should satisfy the following:

$$\alpha_{ba} = \alpha_{ab}, \qquad \Gamma(\alpha_{ab}) = \alpha_{ac}A^c_b + \alpha_{bc}A^c_a,$$

$$\alpha_{ac}\Phi^c_b = \alpha_{bc}\Phi^c_a, \qquad \partial\alpha_{ab}/\partial\dot{x}^c = \partial\alpha_{ac}/\partial\dot{x}^b.$$

These are the Helmholtz conditions.

These conditions provide, in a sense, a solution to the inverse problem of the calculus of variations (for finitely many degrees of freedom), which asks for the circumstances under which a system of second-order ordinary differential equations is of the Euler–Lagrange type. But it is not a completely satisfactory solution since it involves the extraneous functions α_{ab}, when one would hope for conditions on the functions f^a alone. However, the complexity of even the two degrees of freedom case, as analysed by Douglas (1941), makes it clear that such a hope is optimistic in the extreme. Nevertheless, recent work of Henneaux (1982), Sarlet (1982) and others has shown that progress can be made in unravelling the Helmholtz conditions. In particular, Sarlet was able to show that they may be replaced by an equivalent set of conditions, for a matrix-valued function of x^a and $\dot{x}^a$ alone, in which the 'propagation equation' $\Gamma(\alpha_{ab}) = \alpha_{ac}A^c_b + \alpha_{bc}A^c_a$ is replaced by a sequence, in principle infinite, of purely algebraic conditions similar to the condition $\alpha_{ac}\Phi^c_b = \alpha_{bc}\Phi^c_a$ in form, this (at a fixed time) being indeed the first. It appears that in effect Sarlet's conditions are 'initial conditions' which must be satisfied in order that the multiplier matrix, assumed to satisfy $\alpha_{ba} = \alpha_{ab}$, $\alpha_{ac}\Phi^c_b = \alpha_{bc}\Phi^c_a$, and $\partial\alpha_{ab}/\partial\dot{x}^c = \partial\alpha_{ac}/\partial\dot{x}^b$ for some particular value of t, and propagated according to the remaining Helmholtz conditions, will continue to satisfy these conditions for all t.

The general subject area of Lagrangian dynamics is one in which analytical and geometrical approaches usefully complement and inform each other. Sarlet's work was mainly analytical; Henneaux on the other hand used geometrical methods to obtain results which, though analogous in many ways, were less detailed than Sarlet's. One of the authors of the present paper has shown (Crampin 1981, 1983) that by using the geometrical structure of the tangent bundle of a differentiable manifold one can express the Helmholtz conditions in a natural geometrical form, and give a geometrical interpretation of much of Sarlet's analysis. However, this approach, since it is based on tangent bundle geometry, can deal with the time-independent case only. It is clear that a complete understanding, in geometrical terms, of the problem can be achieved only if the time is explicitly included as a coordinate in the space under consideration; that is, only if the underlying manifold is taken to be the evolution space of the system, rather than just the space of configurations and generalised velocities (that is, the tangent bundle over configuration space). Unfortunately, these two spaces have rather different geometrical structures. At the simplest level this is just a consequence of the fact that one of them is odd dimensional, the other even. Thus on evolution space the vector field which defines the dynamics of a Lagrangian system is a characteristic vector field of a 2-form which defines a contact structure; whereas the dynamics of a time-independent system is represented on the tangent bundle over configuration space by a vector field determined by the energy function via a symplectic 2-form. It is a consequence of these differences that geometrical structures on one type of space cannot be simply transported lock, stock and barrel to the other.

The purpose of the present paper is to develop the machinery necessary to carry out the kind of geometrical study undertaken by Crampin (1981, 1983), but in the context of evolution space, for time-dependent systems. It is hoped that eventually a complete understanding of Sarlet's results will be achieved by these means; however, the present paper goes only so far as providing a geometrical equivalent to the Helmholtz conditions in the time-dependent case, which parallels the one given in the time-independent case by Crampin (1981). The presentation of this result, and the description of the appropriate geometrical structures, is preceded by a brief discussion of the nature of evolution space itself.

2. Evolution space

We shall be dealing throughout with a dynamical system with finitely many (say m) degrees of freedom; we therefore take its configuration space to be a differentiable manifold M of dimension m, with local coordinates (x^a). The evolution space of the system is then $R \times TM$. However, we should like to describe the space in a little more detail, partly in order to show how it links up with the spaces appropriate to field theory, and partly to motivate the choices of some of the geometrical structures which we display in § 3.

In fact we wish to think of evolution space as the so-called first jet bundle of (smooth) maps $R \to M$, that is, $J^1(R, M)$. This we shall briefly describe. The points of $J^1(R, M)$ are defined in terms of smooth maps $R \to M$, that is, curves in M. However, we wish to identify the parameter on any curve which is a possible trajectory of the system with the time, considered as an additional coordinate. Time may be incorporated by forming the manifold $R \times M$; then any curve σ in M defines another curve in $R \times M$, its graph, given by $t \to (t, \sigma(t))$. (Alternatively, one may consider $R \times M$ as a (trivial) fibre bundle over R; then σ determines a section of the projection $R \times M \to R$.) We now define, for each fixed $t \in R$, an equivalence relation on curves defined near t, by setting ρ and σ equivalent if $\sigma(t) = \rho(t)$ and $\dot{\sigma}(t) = \dot{\rho}(t)$, where $\dot{\sigma}(t)$ is the tangent vector to σ at t. Thus ρ and σ are equivalent at t if they pass through the same point and have the same tangent vector there. The set of all equivalence classes has the structure of a differentiable manifold of dimension $2m+1$. We call the equivalence class of σ at t its 1-jet and denote this by $j_t^1(\sigma) \in J^1(R, M)$. Each 1-jet may be identified with a triple (t, x, u) where $t \in R$, $x \in M$ and $u \in T_xM$. Thus $J^1(R, M)$ may be identified with $R \times TM$. On the other hand, it is clear that this construction is a particular case of the general construction of the manifold of 1-jets of smooth maps between two manifolds; and in the case when one of these manifolds is space–time and the other a vector bundle over it, and the maps concerned are sections, then the 1-jet bundle is the appropriate space for the study of first-order field theory. (See, for example, Sniatycki (1970).)

The 1-jet bundle $J^1(R, M)$ is fibred over $R \times M$, where the projection map π is given by

$$\pi(j_t^1(\sigma)) = (t, \sigma(t)).$$

It is in fact a vector bundle whose fibre over (t, x) is just T_xM. (Indeed, for what it is worth, $J^1(R, M)$ may also be regarded as the pull-back of TM to $R \times M$ by the map $R \times M \to M$ given by projection onto the second factor.) We shall use coordinates (t, x^a, u^a) on $J^1(R, M)$ defined as follows: (t, x^a, u^a) are the coordinates of the 1-jet at t of the curve in M given in terms of coordinates (x^a) on M by $s \to (x^a + su^a)$.

Any curve σ on M defines a section $j^1(\sigma)$ of the fibration $J^1(R, M) \to R$ (obtained by composing π with projection of $R \times M$ onto the first factor) by

$$j^1(\sigma)(t) = j_t^1(\sigma).$$

Not every section of $J^1(R, M) \to R$ has this particular form, however. In terms of coordinates (t, x^a, u^a), a section $t \to (t, \sigma^a(t), u^a(t))$ will be the 1-jet of the curve σ if and only if

$$u^a(t) = \dot{\sigma}^a(t) \qquad \text{for all } t \text{ in the domain of } \sigma.$$

This condition may be expressed conveniently in terms of the m local contact 1-forms

θ^a, which have the coordinate expression

$$\theta^a = \mathrm{d}x^a - u^a\,\mathrm{d}t;$$

for a section ζ of $J^1(R, M) \to R$ is the 1-jet of a curve in M if and only if

$$\zeta^*\theta^a = 0, \qquad a = 1, 2, \ldots, m.$$

Moreover, any such section ζ of $J^1(R, M) \to R$ must satisfy

$$\zeta^*\,\mathrm{d}t = \mathrm{d}t.$$

It follows that any vector field Γ on $J^1(R, M)$ whose integral curves are all 1-jets of curves in M must satisfy

$$\langle \Gamma, \theta^a \rangle = 0, \qquad a = 1, 2, \ldots, m; \qquad \langle \Gamma, \mathrm{d}t \rangle = 1.$$

We call such a vector field a second-order differential equation field. It takes the form

$$\Gamma = \partial/\partial t + u^a\,\partial/\partial x^a + f^a\,\partial/\partial u^a$$

in terms of coordinates, for some smooth local functions f^a on $J^1(R, M)$. Its integral curves with initial t coordinate 0 have t for parameter and satisfy

$$\dot{x}^a(t) = u^a(t), \qquad \dot{u}^a(t) = f^a((t, x(t)), u(t)).$$

They are thus the 1-jets of the solution curves of the second-order differential equations

$$\ddot{x}^a = f^a(t, x, \dot{x}).$$

3. Geometrical structures on $J^1(R, M)$

We have seen that $J^1(R, M)$ is a vector bundle over $R \times M$ and is equipped with a system of 1-forms, the contact forms, with local basis $\theta^a = \mathrm{d}x^a - u^a\,\mathrm{d}t$. We now describe some of the other geometrical structures on $J^1(R, M)$ which will be useful in this and subsequent work.

Let X be a vector field on $R \times M$; there is a unique vector field $X^{(1)}$ on $J^1(R, M)$, its first prolongation, such that

$$\pi_* X^{(1)} = X,$$

$\mathscr{L}_{X^{(1)}}\theta^a$ is a linear combination of the basic contact 1-forms.

In coordinates, if

$$X = \tau\frac{\partial}{\partial t} + \xi^a\frac{\partial}{\partial x^a} \qquad (\tau, \xi^a \text{ local functions on } R \times M)$$

then

$$X^{(1)} = \tau\,\partial/\partial t + \xi^a\,\partial/\partial x^a + \eta^a\,\partial/\partial u^a$$

where

$$\eta^a = \dot{\xi}^a - u^a\dot{\tau} = u^b\frac{\partial \xi^a}{\partial x^b} + \frac{\partial \xi^a}{\partial t} - u^a\left(u^b\frac{\partial \tau}{\partial x^b} + \frac{\partial F\tau}{\partial t}\right).$$

For any two vector fields X, Y on $R \times M$,

$$[X^{(1)}, Y^{(1)}] = [X, Y]^{(1)}.$$

Moreover,

$$(\partial/\partial t)^{(1)} = \partial/\partial t.$$

It is a straightforward consequence of this definition that for any vector field X on $R \times M$ and any second-order differential equation field Γ, the vector field

$$V = \mathscr{L}_{X^{(1)}}\Gamma - \dot{\tau}\Gamma$$

is vertical (where $\tau = \langle X, \mathrm{d}t\rangle$). Now the addition of a vertical vector field to a second-order differential equation field leads to a new second-order differential equation field. Thus, roughly speaking, prolongations preserve second-order differential equation fields: the effect of the action of the flow of $X^{(1)}$ on Γ is to transform it into a new second-order differential equation field, with a change of parametrisation if $\dot{\tau} \neq 0$.

A vector at a point of $J^1(R, M)$ is said to be vertical if it is tangent to the fibre of $\pi: J^1(R, M) \to R \times M$. The vector fields $V_a = \partial/\partial u^a$ form a local basis of vertical vector fields. We define a set of m local vector fields H_a, in terms of a given second-order differential equation field Γ, by

$$H_a = \partial/\partial x^a - A_a^b\, \partial/\partial u^b, \qquad \text{where } A_a^b = -\tfrac{1}{2}\partial f^b/\partial u^a$$

(and $\Gamma = \partial/\partial t + u^a\, \partial/\partial x^a + f^a\, \partial/\partial u^a$). We call elements of the vector field system spanned by the H_a horizontal. The vector fields $\{H_a, V_a, \Gamma\}$ form a local vector field basis on $J^1(R, M)$, with dual basis of 1-forms $\{\theta^a, \psi^a, \mathrm{d}t\}$ where

$$\psi^a = \mathrm{d}u^a - f^a\, \mathrm{d}t + A_b^a\theta^b = A_b^a \mathrm{d}x^b + \mathrm{d}u^a - (f^a + A_b^a u^b)\, \mathrm{d}t.$$

We shall discuss the provenance and the utility of the horizontal vector fields shortly; but first we list a number of useful relations involving them:

$$[H_a, V_b] = -\tfrac{1}{2}(\partial^2 f^c/\partial u^a \partial u^b)\, V_c = [H_b, V_a],$$

$$[\Gamma, H_a] = A_a^b H_b + \Phi_a^b V_b, \qquad [\Gamma, V_a] = -H_a + A_a^b V_b.$$

Moreover, evidently

$$[V_a, V_b] = 0;$$

on the other hand $[H_a, H_b]$ is not zero in general, but it is always vertical.

We define a type (1, 1) tensor field S on $J^1(R, M)$ by

$$S = V_a \otimes \theta^a.$$

It has the following properties, which indeed determine it uniquely:

S vanishes on vertical vectors, and on second-order differential equation fields;

for any vector field Z on $J^1(R, M)$, $S(Z)$ is vertical;

$S(\partial/\partial t) = -\Delta$, the dilation field on $J^1(R, M)$ considered as a vector bundle over $R \times M$.

Given any vector field X on $R \times M$ we define its vertical lift X^{v} to $J^1(R, M)$ by

$$X^{\mathrm{v}} = S(X^{(1)}).$$

In coordinates, if $X = \tau\, \partial/\partial t + \xi^a\, \partial/\partial x^a$, then

$$X^{\mathrm{v}} = (\xi^a - u^a\tau)\, V_a.$$

The complete and horizontal lifts of X relative to a second-order differential equation field Γ, X^{c} and X^{h} respectively, are defined by

$$X^{\mathrm{c}} = X^{(1)} - \langle X, \mathrm{d}t\rangle\Gamma \qquad \text{and} \qquad X^{\mathrm{h}} = \tfrac{1}{2}([X^{\mathrm{v}}, \Gamma] + X^{\mathrm{c}}).$$

In coordinates

$$X^{\mathrm{c}} = (\xi^a - u^a\tau)\, \partial/\partial x^a + (\dot{\xi}^a - u^a\dot{\tau} - f^a\tau)\, \partial/\partial u^a$$

and

$$X^{\mathrm{h}} = (\xi^a - u^a\tau) H_a.$$

(Compare with Crampin (1983).) Note that X^{c} and $X^{(1)}$ coincide for vector fields X on M. The following relations hold (where $\tau = \langle X, \mathrm{d}t\rangle$ and $\rho = \langle Y, \mathrm{d}t\rangle$):

$$[X^{\mathrm{v}}, Y^{\mathrm{v}}] = \tau Y^{\mathrm{v}} - \rho X^{\mathrm{v}}, \qquad [X^{\mathrm{v}}, Y^{(1)}] = [X, Y]^{\mathrm{v}} + \dot{\tau} Y^{\mathrm{v}} - \dot{\rho} X^{\mathrm{v}},$$

$$\mathscr{L}_{X^{(1)}} S = \dot{\tau} S, \qquad \mathscr{L}_{X^{\mathrm{v}}} S = \tau S - X^{\mathrm{v}} \otimes \mathrm{d}t.$$

Now let Γ be any second-order differential equation field on $J^1(R, M)$. We establish some results about $\mathscr{L}_\Gamma S$. (Compare Crampin (1983) for the time-independent case.)

Lemma 1.

$$(\mathscr{L}_\Gamma S)(\Gamma) = 0.$$

Proof.

$$(\mathscr{L}_\Gamma S)(\Gamma) = [\Gamma, S(\Gamma)] - S([\Gamma, \Gamma]).$$

Lemma 2. For any vector field X on $R \times M$,

$$(\mathscr{L}_\Gamma S)(X^{\mathrm{v}}) = X^{\mathrm{v}}.$$

Proof.

$$\begin{aligned}(\mathscr{L}_\Gamma S)(X^{\mathrm{v}}) &= [\Gamma, S(X^{\mathrm{v}})] - \mathrm{S}([\Gamma, X^{\mathrm{v}}]) \\ &= S([X^{\mathrm{v}}, \Gamma]) \\ &= [X^{\mathrm{v}}, S(\Gamma)] - (\mathscr{L}_{X^{\mathrm{v}}} S)(\Gamma) \\ &= X^{\mathrm{v}}.\end{aligned}$$

Lemma 3. For any vector field X on $R \times M$,

$$(\mathscr{L}_\Gamma S)(X^{\mathrm{h}}) = -X^{\mathrm{h}}.$$

Proof. By definition,

$$X^{\mathrm{h}} = \tfrac{1}{2}([X^{\mathrm{v}}, \Gamma] + X^{\mathrm{c}}).$$

Note that

$$S(X^{\mathrm{c}}) = S(X^{(1)}) = X^{\mathrm{v}}.$$

Thus

$$(\mathscr{L}_\Gamma S)(X^{\mathrm{c}}) = [\Gamma, X^{\mathrm{v}}] - S([\Gamma, X^{\mathrm{c}}]).$$

Now $[\Gamma, X^c]$ is vertical, since, as we pointed out above,

$$[X^{(1)}, \Gamma] = \dot{\tau}\Gamma + V$$

where V is vertical, and thus

$$[X^c, \Gamma] = [X^{(1)}, \Gamma] - [\tau\Gamma, \Gamma] = \dot{\tau}\Gamma + V - \dot{\tau}\Gamma = V.$$

So in fact

$$(\mathscr{L}_\Gamma S)(X^c) = -[X^v, \Gamma].$$

On the other hand, as we showed in the proof of lemma 2,

$$S([X^v, \Gamma]) = X^v = S(X^c);$$

so $[X^v, \Gamma] - X^c$ lies in the kernel of S; and since

$$\langle [X^v, \Gamma], \mathrm{d}t \rangle = \langle X^c, \mathrm{d}t \rangle = 0$$

it is in fact vertical. Thus, by using lemma 2 again, we obtain

$$\begin{aligned}(\mathscr{L}_\Gamma S)(X^h) &= (\mathscr{L}_\Gamma S)(\tfrac{1}{2}([X^v, \Gamma] - X^c) + X^c) \\ &= \tfrac{1}{2}([X^v, \Gamma] - X^c) - [X^v, \Gamma] \\ &= -X^h.\end{aligned}$$

Thus $\mathscr{L}_\Gamma S$ has eigenvalues 0, +1 and −1. The eigenspace at a point of $J^1(R, M)$ corresponding to the eigenvalue 0 is spanned by Γ, while the eigenspaces corresponding to the eigenvalues +1 and −1 are the vertical and horizontal subspaces respectively. By reversing the direction of this argument one may give a coordinate free definition of the horizontal subspaces: they are the eigenspaces of $\mathscr{L}_\Gamma S$ corresponding to the eigenvalues −1.

The 1-form basis $\{\theta^a, \psi^a, \mathrm{d}t\}$ adapted to this direct sum decomposition of each tangent space to $J^1(R, M)$ turns out to be a very convenient one to use in the context of Lagrangian dynamics and the inverse problem. In part, this is because

$$\langle \Gamma, \theta^a \rangle = \langle \Gamma, \psi^a \rangle = 0, \qquad \langle \Gamma, \mathrm{d}t \rangle = 1.$$

But of course this particular basis is not the only one to have these properties. Indeed, it might seem that a more obvious choice would be $\{\theta^a, \mathrm{d}u^a - f^a\,\mathrm{d}t, \mathrm{d}t\}$, a 1-form basis which in fact has been frequently used before (for example, by Crampin (1977) and Prince (1983)). The additional advantages of using the new basis will become apparent below.

The horizontal distribution is a generalisation of the one defined by a symmetric connection, which is recovered when Γ is the geodesic spray of the connection. This approach to questions concerned with geodesic conservation laws in general relativity has been exploited by Prince and Crampin (1983).

4. The Euler–Lagrange field and the Cartan form

Suppose we are given a Lagrangian, that is, a smooth function L on $J^1(R, M)$. It determines, in the familiar way, a second-order differential equation field Γ, whose integral curves are the 1-jets of the solutions curves of the Euler–Lagrange equations.

We assume that the Lagrangian is regular, which is to say that (in terms of local coordinates) the matrix whose entries are

$$\alpha_{ab} = \partial^2 L/\partial u^a \partial u^b = V_a(V_b(L))$$

is non-singular. Then Γ, which we call the Euler–Lagrange field, is given by

$$\Gamma = \partial/\partial t + u^a\, \partial/\partial x^a + f^a\, \partial/\partial u^a,$$

where

$$\alpha_{ab} f^b = \partial L/\partial x^a - u^b\, \partial^2 L/\partial x^b \partial u^a - \partial^2 L/\partial u^a \partial t.$$

Moreover, (α_{ab}) satisfies the Helmholtz conditions. Since $\{H_a, V_a, \Gamma\}$ and $\{\theta^a, \psi^a, \mathrm{d}t\}$ are dual local bases, Γ is uniquely determined by the equations

$$\langle \Gamma, \theta^a \rangle = 0, \qquad \langle \Gamma, \psi^a \rangle = 0, \qquad \langle \Gamma, \mathrm{d}t \rangle = 1.$$

As an alternative to all but the last of these, one may replace the $2m$ conditions involving 1-forms with one condition involving a 2-form, as follows: if we define a 2-form Ω by

$$\Omega = \alpha_{ab} \psi^a \wedge \theta^b$$

then Γ is the unique characteristic vector field of Ω satisfying the final normalisation condition: that is, Γ satisfies

$$i_\Gamma \Omega = 0, \qquad \langle \Gamma, \mathrm{d}t \rangle = 1$$

and is unique in so doing. It is evident that Γ does satisfy these conditions; also

$$\Omega \wedge \Omega \wedge \ldots \wedge \Omega = \pm \det(\alpha_{ab}) \psi^1 \wedge \ldots \wedge \psi^m \wedge \theta^1 \wedge \ldots \wedge \theta^m \neq 0$$

(there being m factors on the left-hand side) since (α_{ab}) is non-singular and $\psi^1, \ldots, \psi^m$, $\theta^1, \ldots, \theta^m$ are linearly independent, and this implies that the space of characteristic vectors of Ω at each point of $J^1(R, M)$ is one-dimensional.

Again, many 2-forms could be found with this property. What is significant about this one, and therefore significant about the particular 1-forms involved in its construction, is that Ω is actually the exterior derivative of the Cartan 1-form

$$\theta_L = L\, \mathrm{d}t + \mathrm{d}L \circ S = L\, \mathrm{d}t + (\partial L/\partial u^a)\theta^a.$$

This we now prove, with the aid of a lemma.

Lemma 4. The vertical vector fields and the horizontal vector fields determined by the Euler–Lagrange field of L satisfy

$$V_b H_a(L) = V_a H_b(L).$$

Proof. The Euler–Lagrange equations may be written

$$\Gamma V_a(L) = \partial L/\partial x^a.$$

Now, by definition,

$$H_a = \tfrac{1}{2}([V_a, \Gamma] + \partial/\partial x^a),$$

since the complete lift of $\partial/\partial x^a$ from $R\times M$ to $J^1(R,M)$ is just $\partial/\partial x^a$. Thus

$$\begin{aligned} V_a\Gamma(L) &= [V_a,\Gamma](L)+\Gamma V_a(L) \\ &= 2H_a(L)-\partial L/\partial x^a+\Gamma V_a(L) \\ &= 2H_a(L). \end{aligned}$$

Consequently,

$$V_bH_a(L)=\tfrac{1}{2}V_bV_a(\Gamma(L)),$$

and so, since $[V_a, V_b]=0$,

$$V_bH_a(L)=V_aH_b(L).$$

Theorem 1.

$$\alpha_{ab}\psi^a\wedge\theta^b=\mathrm{d}(L\,\mathrm{d}t+\mathrm{d}L\circ S)=\mathrm{d}\theta_L.$$

Proof. We evaluate $\mathrm{d}\theta_L$ by using the formula

$$\mathrm{d}\theta_L(Y,Z)=Y\langle Z,\theta_L\rangle-Z\langle Y,\theta_L\rangle-\langle[Y,Z],\theta_L\rangle$$

with appropriate choices of vector fields from $\{H_a, V_a,\Gamma\}$ to stand for Y, Z in turn. Note that

$$\langle H_a,\theta_L\rangle=V_a(L),\qquad \langle V_a,\theta_L\rangle=0,\qquad \langle\Gamma,\theta_L\rangle=L.$$

Thus, since $[H_a, H_b]$ is vertical,

$$\begin{aligned} \mathrm{d}\theta_L(H_a,H_b) &= H_a(V_b(L))-H_b(V_a(L)) \\ &= [H_a, V_b](L)-[H_b, V_a](L)+V_b(H_a(L))-V_a(H_b(L)) \\ &= 0, \end{aligned}$$

using the lemma, and the fact that $[H_a, V_b]=[H_b, V_a]$. Clearly,

$$\mathrm{d}\theta_L(V_a, V_b)=0.$$

Moreover,

$$\mathrm{d}\theta_L(\Gamma,H_a)=\Gamma(V_a(L))-H_a(L)-A_a^b V_b(L)=0$$

by virtue of the Euler–Lagrange equations, while

$$\mathrm{d}\theta_L(\Gamma, V_a)=-V_a(L)+\langle H_a,\theta_L\rangle=0.$$

Finally,

$$\mathrm{d}\theta_L(V_a,H_b)=V_a(V_b(L))=\alpha_{ab},$$

since $[V_a, H_b]$ is vertical.

The significance of the 1-form basis $\{\theta^a, \psi^a, \mathrm{d}t\}$ lies in the fact that with respect to it the Cartan 2-form has this particularly simple expression. Consider now the inverse problem. The 1-forms ψ^a are defined by the given second-order differential equation field, of course; thus 2-forms Ω with the algebraic properties enjoyed by a Cartan 2-form may easily be constructed. But in order that such a 2-form Ω be a Cartan 2-form, it must also satisfy a differential condition, or conditions, equivalent to closure

($d\Omega = 0$). We show in § 5 that if Ω satisfies the appropriate algebraic and differential conditions then it takes the form $\Omega = \alpha_{ab}\psi^a \wedge \theta^b$, where the functions α_{ab} are the elements of a multiplier matrix and satisfy the Helmholtz conditions.

5. The Helmholtz conditions

We have shown that for an Euler–Lagrange field, if we set $\Omega = \alpha_{ab}\psi^a \wedge \theta^b$ then Ω satisfies the following conditions:

$\Omega(V_1, V_2) = 0$ for any vertical vector fields V_1, V_2,

$i_\Gamma\Omega = 0$,

$d\Omega = 0$.

Note that the second two of these conditions entail that $L_\Gamma\Omega = 0$.

We now prove a converse to this result, namely that a 2-form Ω satisfying these conditions for a given second-order differential equation field Γ must necessarily take the form $\alpha_{ab}\psi^a \wedge \theta^b$ where the α_{ab} satisfy the Helmholtz conditions. In fact the final condition $d\Omega = 0$ may be considerably weakened. The theorem below is the analogue for time-dependent dynamical systems of the results of Crampin (1981) for the time-independent case.

Theorem 2. Given a second-order differential equation field Γ, the necessary and sufficient condition for there to be a Lagrangian for which Γ is the Euler–Lagrange field is that there should exist a 2-form Ω, of maximal rank, such that

$\Omega(V_1, V_2) = 0$ for any pair of vertical vector fields V_1, V_2,

$i_\Gamma\Omega = 0$,

$L_\Gamma\Omega = 0$,

$i_H d\Omega(V_1, V_2) = 0$ for any horizontal vector field H and any pair of vertical vector fields V_1, V_2.

Proof. The necessity is apparent. To prove sufficiency, we show that these conditions imply that $\Omega = \alpha_{ab}\psi^a \wedge \theta^b$ where the α_{ab} satisfy the Helmholtz conditions.

The first two (purely algebraic) conditions imply that when expressed in terms of the basis 1-forms $\{\theta^a, \psi^a, dt\}$, Ω has the form

$$\Omega = \alpha_{ab}\psi^a \wedge \theta^b + \beta_{ab}\theta^a \wedge \theta^b.$$

From the assumption that Ω has maximal rank, so that the m-fold exterior product $\Omega \wedge \Omega \wedge \ldots \wedge \Omega \neq 0$, it follows that (α_{ab}) is non-singular. We may assume $\beta_{ba} = -\beta_{ab}$.

Next, we compute $\mathscr{L}_\Gamma\Omega$. By dualising the formulae for $[\Gamma, H_a]$ and $[\Gamma, V_a]$ given in § 3 one obtains

$$\mathscr{L}_\Gamma\theta^a = -A^a_b\theta^b + \psi^a, \qquad \mathscr{L}_\Gamma\psi^a = -\Phi^a_b\theta^b - A^a_b\psi^b.$$

Thus

$$\mathscr{L}_\Gamma\Omega = \Gamma(\alpha_{ab})\psi^a \wedge \theta^b - \alpha_{bc}(\Phi^c_a\theta^a + A^c_a\psi^a) \wedge \theta^b + \alpha_{ac}\psi^a \wedge (\psi^c - A^c_b\theta^b)$$
$$+ \Gamma(\beta_{ab})\theta^a \wedge \theta^b + 2\beta_{ab}(\psi^a - A^a_c\theta^c) \wedge \theta^b.$$

So $\mathscr{L}_\Gamma\Omega = 0$ only if

(i) $$\Gamma(\beta_{ab}) = \beta_{ac}A^c_b - \beta_{bc}A^c_a + \alpha_{ac}\Phi^c_b - \alpha_{bc}\Phi^c_a,$$

(ii) $\Gamma(\alpha_{ab}) = \alpha_{ac}A^c_b + \alpha_{bc}A^c_a + 2\beta_{ab}$,

(iii) $\alpha_{ab} = \alpha_{ba}$.

From (ii), by taking the skew-symmetric part it follows that $\beta_{ab} = 0$. We thus obtain the first three Helmholtz conditions.

Finally, the condition $i_H\, d\Omega(V_1, V_2) = 0$ ensures that the coefficient of $\psi^b \wedge \psi^c$ in $i_H d\Omega$ vanishes; when $H = H_a$ this turns out to be $\partial\alpha_{ab}/\partial u^c - \partial\alpha_{ac}/\partial u^b$, and so the final Helmholtz condition is obtained.

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